

Prizes, Groups and Pivotal Voting in a Poisson Voting Game

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Abstract

We model elections between two parties in a Poisson random population of voters (Myerson 1998, 2000). In addition to offering different policy benefits, parties offer contingent prizes to those identifiable groups of voters that offer the highest level of political support. In large populations, voters are only likely to influence the electoral outcome when the vote share between two parties is perfectly equal and even then their influence on the outcome is small. In contrast voters retain significant influence over the distribution of prizes even in lopsided elections. Equilibrium behavior is driven by voters competing to win preferential treatment for their group and not by policy concerns. The results address variance in turnout in elections, political rewards and the persistence of dominant parties even when they are popularly perceived as inferior.

INTRODUCTION

We present a random population model of voting and examine the consequences of parties targeting selective rewards to those groups of voters that offer them the highest level of electoral support. The model explains voting behavior in both competitive electoral systems, such as those seen in established democracies, and patronage styled democracies in which a dominant party persists in the presence of free and fair elections even when it is widely regarded as inferior to other parties. The model resolves problems seen in standard accounts of rational voting models. In particular, we explain high voter turnout in large electorates in both competitive and non-competitive elections, resolve credibility issues in vote buying accounts within patronage democracies and explain the persistence of dominant parties.

The basic setup integrates Smith and Bueno de Mesquita's (2012) concept of Contingent Prize Allocation Rules (CPAR) and Myerson's (1998, 2000) Poisson random population of voters model. In the simple CPAR examined here, parties provide selective rewards (a.k.a. prizes) to identifiable groups of voters based on their level of electoral support. One simple illustration of such a grouping is geographical districts. In established democracies parties rarely observe the vote choice of individuals. Yet, they readily observe the level of electoral support at the ward or precinct level. Under the CPAR applied here, parties reward the most supportive group. For instance, in the geographical

context they might allocate new infrastructure projects such as a new school or hospital to the most supportive precinct within a larger electoral district. Alternatively they could reward a particular ward or precinct through the provision of superior services, such as better trash pickup or more reliable street plowing. Parties might also disproportionately hire public employees from the most supportive group. This was a standard practice within the party machines which dominated many large US cities (Allen 1993). Richard J. Daley, the long term mayor of Chicago, was notorious in this regards (Rakove 1975). This is perhaps unsurprising since the internal rules of the Democratic Party of Cook County (which contains Chicago) specifies that on committees, ward representatives are given voting rights in proportion to the level of democratic votes their ward delivered in previous elections (Gosnell 1937). US national parties also structure rules to reward their loyalists. For instance, both parties skew Presidential nomination procedures in favor of states that gave them high levels of support in previous elections (see for instance, Democratic Party Headquarters 2007).

The model examines two forms of prizes: victory prizes, those that a party can deliver only if it is elected to office, and guaranteed prizes which it can deliver to supportive groups even if it does not get elected. The latter type of goods might be important in a Federal style systems where a party could use resources at one level of government to reward voters for their support at another. The mathematical analysis of both prizes is similar so we do not dwell on what aspects of government and party spending fall into each type of prize.

Geographical groupings provide easy exposition of the model. The argument, however, is applicable to any set of groups provided the parties observe the level of electoral support from each group and parties can selectively reward groups. Groups might, therefore, be based on religion, ethnicity or profession instead of geographical voting districts. What matters is that parties can, in the aggregate, observe who supports them and can reward the most supportive groups.

The analysis takes the form of a rational choice vote calculation in which voters individually compare the benefits of voting for each party with the costs of casting a vote (Downs 1957; Riker and Ordeshook 1968). A classic criticism of such an approach, and one of Green and Shapiro's (1996) main arguments in their repudiation of rational choice modeling, is that in a large electorate voters are extremely unlikely to be pivotal. Hence, if voting is costly and their vote is extremely unlikely to matter, then critics questions why voters turnout. We provide a rational choice model in which voters are unlikely to be pivotal in altering who wins and yet still turn out to vote. Hence, the model provides a solution to the voter-turnout puzzle.

The model considers three type of pivot probability. Consistent with standard models, we examine a voter's influence on which party wins the election. We refer to this as the outcome pivot. As Schwartz (1987) argues, vote choice might matter at a sub constituency level. If parties reward supportive groups, then an individual voter's vote choice affects the distribution of two kinds of prizes. The extent to which an additional vote changes the distribution of prizes is referred to as the guaranteed prize pivot or the victory prize pivot, depending upon whether the party can always distribute the prize or needs to win the election in order to do so.

Smith and Bueno de Mesquita (2012) illustrate how voters retain their prize pivotalness even in large electorates with a stylized example of three villages each with n voters. The victorious party offers to build a hospital, or other project, in the village that gives it the most votes. There is an equilibrium in which all voters support one party and the pivotalness of the vote choice is $1/3$ even if the electorate is large. In this illustration, no single voter is influential in affecting which party wins. Yet, by voting for the dominant party each voter gives their village a $1/3$ chance of receiving the prize. If they abstain or vote for another party, then their village has one fewer votes than the other villages and so their village has no chance of receiving the prize. Provided that the cost of voting is less than $1/3$ of the value of the prize, all voters strictly want to support the dominant party. Of course, their example is a little contrived, but the basic ideal of prize allocation holds in much wider generality.

We introduce two forms of uncertainty. First, voters do not have perfect information about even the basic parameters of the voting system, namely the exact population size. We treat the population size (and therefore the population of each group) as a random variable. Given this assumption, no-one is quite certain how many voters there are in each group. Second, each voter has a personal -and private- evaluation of one party relative to the other, which we model as a random variable. We model the population size as a Poisson random variable (to be explained below), but we put only the bare minimum assumptions on the probability distribution describing voters' preferences.

Myerson (1998, 2000) shows that treating population size as a Poisson random variable creates a flexible framework within which it is straightforward to analyze pivotal voting decisions. For instance, he shows how the Poisson approach avoids the messy combinatoric calculations involved in large fixed population voting models (Palfrey and Rosenthal 1983, 1985; Ledyard 1984). Consistent with these models he finds that in large electorates, voters are only outcome pivotal in very close elections and even then their influence is small; so even a small cost of voting discourages significant turnout. The key to the analyses presented here is that while outcome pivotalness becomes negligible in a large electorate, prize pivotalness goes to zero much more slowly. In particular as N , the expected size of the population, increases prize pivotalness is proportional to $\frac{1}{\sqrt{N}}$, so it decays slowly in terms of population size. In contrast, outcome pivot is proportional to $\frac{1}{\sqrt{N}}e^{-N(\sqrt{p}-\sqrt{q})^2}$, where p and q are the probabilities that voters support parties A and B respectively. Hence, except in the precise case of perfect electoral balance ($p = q$), outcome pivot declines at an exponential rate as the expected population size increases. Therefore, except in extremely close elections, prize pivots come to dominate outcome pivot as the electorate grows. We formally develop the concept of prize pivotalness and derive approximations for the extent to which voters influence electoral outcomes and the distribution of prizes in large electorates of uncertain size. The central result is that the primary motivation to vote is competition for selective rewards rather than the policy differences between parties. We explore these ideas in both competitive elections, in which parties are anticipated to receive roughly similar numbers of votes, and in non-competitive elections, where

one party dominates. Prizes greatly increase turnout in both settings.

Dispensing prizes can result in stable patronage-style arrangements in which one party predominates and virtually always wins. Voters do not vote in the hope of influencing the electoral outcome. Rather, they vote to increase their group's chance of receiving a prize. We examine these dominant party equilibria in two settings. First, we consider symmetric equilibria in which all groups support parties at the same rate. Following that, we explore equilibria in which different groups support the parties at different rates. Effective the groups polarize, such that some groups support party A while other groups predominantly direct their votes to the alternative party. Such polarization can result in a stable electoral arrangement where one party virtually always wins. The coordination of votes into loyalist and opposition groups is an equilibrium feature of individual voters maximizing their welfare. Our final analysis shows that while the presence of prizes targeted towards the most supportive group increases voter turnout in loyalist groups, it simultaneously suppresses turnout in moderate groups.

LITERATURE REVIEW

Pivotality lies at the heart of rational choice models of voting behavior (Downs, 1957; Aldrich 1993; Riker and Ordeshook 1968; Ferejohn and Fiorina 1974). Voters not only assess their expected rewards under each party, they also factor in the likelihood that their vote matters. Formally, a vote only matters if it breaks a tie or turns defeat into a tie. Under all other circumstances, an additional vote is immaterial in determining which party wins. In a large electorate, even if the outcome is expected to be close, the probability that a voter's vote matters is extremely small (Myerson 1998, 2000). Although Myatt (2011), building off an earlier result by Good and Mayer (1975), suggests that in the presence of uncertainty about the relative popularity of the parties, pivot probabilities do not go to zero as quickly as in perfect information models, the likelihood of influencing an electoral outcome is small. Given this minuscule influence on outcomes, most voters are expected to abstain. That electoral turnout is much higher than anticipated by such approaches has been a central critique of rational choice models (Green and Shapiro 1996; see Geys 2006 for a survey of this literature). That voting does not matter in terms of who wins is particularly pertinent in the case of electoral systems with a dominant party that continually wins.

Many branches of the voting literature consider factors beyond pure policy comparisons of parties. For instance, voters are motivated by personal or local benefits, such as patronage and pork (Ferejohn 1974; Fenno 1978; Schwartz 1987; Stokes 2005, 2007). Most relevant to the discussion here is Schwartz's (1987) expected utility model in which he argues that voters care about how their precinct votes in terms of potentially courting favor from the victorious party. He shifts the focus of turnout from the global policy difference between parties to the selective provision of local public goods or club goods to sub-electorates. Although he maintains a focus on pivotality with regard to which party wins, his approach concentrates on a smaller local level of analysis where

voters are likely to be more influential than at the macro level. Our analysis focuses on pivotality with regards to the distribution of prizes.

The vote buying literature assesses where parties can most effectively buy electoral support (Ansolabehere and Synder. 2006; Myerson 1993, Dekel et al 2008, Kovenock and Roberson 2009, Cox and McCubbins 1986, Lindbeck and Weibull 1987, and Dixit and Londregan 1995, 1996). One common question raised is whether parties increase their vote share more by offering rewards to party loyalists to increase turnout within these groups or to marginal voters, a swing in whose vote might be critical. Such approaches treat the parties as strategic competitors while the voters respond to rewards in a simple manner. However, consistent with the critique of pivotal voting, although these vote buying tactics increase the attractiveness of one party relative to another, they do not mitigate the problem that individual voters have little influence over electoral outcomes and that, therefore, making a party more attractive only minimally increases the incentive to vote for it. Further, such approaches fail to explain voting behavior when one party is widely anticipated to win.

Pork or patronage rewards are often proffered for voter support (Ferejohn 1974; Fenno 1978; Kitschelt and Wilkinson 2007; Stokes 2005). By offering up-front bribes and the prospects of rewards, such as jobs or better services, after the election, patronage based parties directly influence voters. Even if such targeted rewards might be economically inefficient relative to public goods (Lizzeri and Persico 2001), vote buying is politically valuable because it obfuscate the pivotality issue. Since the quid pro quo of benefits for votes is carried out at the individual level, pivotality does not generally concern patronage vote buying models. Parties buy individual votes rather than make themselves electorally more attractive. As such, voters don't discount the value of the party by the likelihood that their vote is influential. Yet a number of credibility issues surround patronage vote buying (Stokes 2005, 2007).

At least in established democracies, once a voter enters the voting booth, parties can not observe whether the voter delivers the promised vote (although see Gerber et al 2009). Neither can the voter be certain that a party will deliver its promised rewards after the election. Norms and reciprocity are often offered as solutions to these credibility issues (see Kitschelt and Wilson 2007 for a reviews) and scholars such as Stokes have developed repeated play models to explicitly address these concerns (Stokes 2005). However, other problematic issues remain. For example, relatively few voters receive goods from the party. Stokes (2005 p. 315) illustrates the problem with the example of an Argentinean party worker given ten tiny bags of food with which to buy the 40 voters in her neighborhood. Further, survey evidence by Brusco, Nazareno and Stokes (2004) suggests that the receipt of bribes does not guarantee that voters support the party. Similarly, Guterbock (1980) found that Chicago residents who received party service were no more likely to vote Democratic than those receiving no favors.

The contingent prize allocation rule approach places the vote-buying aspects of patronage within the context of pivotality within the whole election. That voters have significant influence over the

distribution of resources provides an equilibrium mechanism for credible voter rewards even though individual votes are unobservable.

BASIC SETUP

There are two political parties A and B that compete for a single office in a winner-takes-all electoral system. The number of voters is treated as a random variable which is Poisson distributed with a mean size of N : the probability that there are a total of θ voters is $\frac{e^{-N} N^\theta}{\theta!}$. The voting population is composed of K roughly equal sized groups, labeled $1, 2, \dots, K$. Each voter in each group votes either for A, for B or abstains. Let the number of votes for parties A and B in group k be represented as a_k and b_k and let the total number of votes for A and B be represented as $a = \sum_{k=1}^K a_k$ and $b = \sum_{k=1}^K b_k$. The number of voters in a group, η_k , is unobserved by the parties or voters but is assumed to be a Poisson distributed random variable with mean N_k : $\Pr(\eta_k = \theta) = \frac{e^{-N_k} N_k^\theta}{\theta!}$. We assume the groups are of roughly similar size: $N_k = N/K$ for all k .

If p_k and q_k represent the probabilities that members of group k vote for parties A and B respectively, then, by the decomposition property of the Poisson distribution, a_k and b_k , the number of votes for each party, are also Poisson distributed with means $N_k p_k$ and $N_k q_k$. Party A wins the election if it receives more votes than party B: $a > b$. In the event of a draw ($a = b$) the outcome of the election is determined by the outcome of a coin flip.

Voters care about both policy benefits and any potential prizes the parties distribute. With regard to policy benefits, voter i receives a policy reward of $\gamma + \varepsilon_i$ if party A wins the election and a policy payoff of 0 if B wins. The γ term represents the average evaluation of party A relative to party B. The random variable ε_i represents individual i 's private evaluation of party A relative to party B; without loss of generality we will assume that ε_i has mean zero. Any average change in preference will be subsumed into the constant γ . We assume the individual evaluations are identically independently distributed with distribution $\Pr(\varepsilon_i < r) = F(r)$. To avoid the need to introduce additional notation for mixed strategies, we assume $F(r)$ is continuous. All the examples are constructed assuming a standard Gaussian distribution.

Contingent Prize Allocation Rule.—

In addition to policy concerns, the voters care about any prizes that the parties might distribute. Following Smith and Bueno de Mesquita (2012), we examine a Contingent Prize Allocation Rule (CPAR) where parties give a reward to the group that delivers the largest number of votes for that party. We consider two types of prizes. First, a guaranteed prize Γ_A that A hands out whether it wins the election or not, and second, a victory prize, Ω_A , that A can only hand out if it wins the election. The corresponding prizes from party B are Γ_B and Ω_B .

The victory prizes can be thought of as the spoils of electoral success. If party A wins the election, then it can choose from which group it hires public employees. It can also selectively provide public services. For instance, it can decide which neighborhoods have their trash picked up or their streets

plowed. It might also decide the location of infrastructure projects or which schools and hospitals to renovate on the basis of political support. Allocating such rewards requires winning office.

Parties might also be able to provide more limited, guaranteed prizes to their political loyalists even if they do not win. These rewards might be especially important in a federal system because a party might reward supporters for voting at one level of government with rewards from another. If, for example, party A controls the state level government it might hire public employees from a particular group to reward them for their support in city level elections. This might be particularly pertinent if A is perceived as having little chance of winning the city level election. Mathematically we assume all members of the group benefit equally from the allocation of the prize. Effectively we treat the prize as a club good or local public good. It benefits members of the group, but not those outside of it. In expectation, private goods such as jobs or transfers might also satisfy this assumption if they are randomly allocated to members within the group.

The contingent prizes, whether guaranteed or victory prizes, are given to the most supportive group. In the event that multiple groups tie in terms of support, each group is given rewards worth half the value of the prize. This latter assumption is clearly a simplification. It might, for instance, seem more reasonable to suppose that in the event of a three-way tie each group gets 1/3 of the prize. This alternative assumption would significantly complicate the mathematics without changing the substantive conclusions. As will be seen in lemma 3 below, the probability of a tie is given by the Skellam distribution and the probability of a three way tie is an order of magnitude less likely than that of a two way tie. Therefore, as the population gets large, the probability of three way or higher ties goes to zero much faster than the probability of two way ties and so the calculation of pivotality is dominated by the influence of two-way ties only. Myerson (2000) formally compares the rate at which the probability of different events goes to zero. The vote calculus derived here comes arbitrarily close to that resulting from more elaborate assumptions. We use the notation that $\hat{a}_{-k} = \max\{a_j : j \neq k\}$ is the largest number of votes for party A by any group except group k .

Party A gives the Guaranteed Prize, Γ_A , to the group that gives it the most votes. Group k 's expected share of A's Guaranteed Prize therefore equals
$$\begin{cases} 1 & \text{if } a_k > \hat{a}_{-k} \\ 1/2 & \text{if } a_k = \hat{a}_{-k} \\ 0 & \text{if } a_k < \hat{a}_{-k} \end{cases} .$$
 The first case corresponds to group k being the unique most supportive group. In the second case group k ties for being the most supportive group and so get a half share of the prize. Finally, if group k is not one of the most supportive groups, then it receives no share of the prize.

Party A can only give out the Victory Prize Ω_A if it wins the election. Group k gets the whole victory prize Ω_A if party A wins the election outright and group k provides the (unique) largest number of votes for party A ($a > b$ & $a_k > \hat{a}_{-k}$). Group k get an expected half prize if either party A gets the same number of votes as party B and group k is one of the strongest supporting groups for A ($a = b$ & $a_k \geq \hat{a}_{-k}$) or if party A gets more votes than party B and group k ties in its maximal

support of party A ($a > b$ & $a_k = \hat{a}_{-k}$). If either A loses the election or another group provides more votes for A, then group k gets no share of the victory prize ($a < b | a_k < \hat{a}_{-k}$). Group k 's expected share of the Victory Prize equals
$$\begin{cases} 1 & \text{if } a_k > \hat{a}_{-k} \text{ and } a > b \\ 1/2 & \text{if } a_k = \hat{a}_{-k} \text{ and } a > b \\ 1/2 & \text{if } a_k \geq \hat{a}_{-k} \text{ and } a = b \\ 0 & \text{if } a_k < \hat{a}_{-k} \text{ or } a < b \end{cases}.$$
 As discussed above, we don't treat three-way ties differently from two-way ties because of their rarity in large populations.

PIVOTALITY AND VOTING

In the voting game, each voter simultaneously decides whether to vote A, vote B or abstain, $\{A, B, \phi\}$. Voting is costly. Any voter who votes, whether for A or B, pays a cost c . The concept of pivotality lies at the heart of rational choice analyses of voting. Voters weigh the costs and benefits of voting: they vote for the alternative they prefer (at least in two-party competition), but they only vote when their expected influence on the outcome outweighs the cost of voting. The standard concept of this influence is the likelihood of shifting the outcome from one party winning to another. We refer to this as the outcome pivot.

Voters can also be pivotal in terms of the distribution of the prize. That is, by voting for party A, a voter not only increases the likelihood that party A wins, she also increases the probability that her group will be the most supportive group and so receive selective benefits from party A. We refer to the likelihood of being pivotal in terms of prize allocation as the Guaranteed Prize Pivot GA with regard to the prize party A gives out whether it wins or not, and the Victory Prize Pivot, VA , with regard to the prize party A allocates only if it wins.

As Myerson (1998) demonstrates, the Poisson model provides an extremely convenient framework for modeling pivotality. The approach assumes that the size of a group is a Poisson distributed random variable. In the case of the current model, group k has η_k members, where η_k is an unknown random variable that is Poisson distributed with mean N_k . Given this Poisson assumption, from the perspective of each member of group k , the votes of the other $\eta_k - 1$ members of k (excluding themselves) can also be assumed to be Poisson distributed with mean N_k . This result, which Myerson (1998, Theorem 2 p384) refers to as *environmental equivalence*, means that each voter's calculations about the other members of the group is mathematically equivalent to an external analyst's perspective of the whole group.

Environmental equivalence results from two factors perfectly offsetting each other. The first factor is a signal about group size. Given that an individual is a member of the group provides the signal that the expected group size is larger than the prior mean. The second factor is that when formulating her optimal actions, a voter considers only the $\eta_k - 1$ other members of the group. These two factors result in each voter's perception of the other members of her group being identical to the analyst's perception of the whole group. This feature makes the Poisson framework especially attractive for modeling pivotality.

The proposition below provides a definition and calculation of pivot probabilities. The following terminology simplifies the statement of the pivots: $w_{kj} = \Pr(a_k > a_j)$, $t_{kj} = \Pr(a_k = a_j)$ and $s_{kj} = \Pr(a_k + 1 = a_j)$ and $u = \Pr(a = b + 2)$, $v = \Pr(a = b + 1)$, $z = \Pr(a > b)$, $x = \Pr(a = b)$ and $y = \Pr(a = b - 1)$.

Proposition 1 *If i is a representative voter in group k then the probability her vote choice alters the electoral outcome and the distribution of prizes is given by the following pivot probabilities.*

Outcome Pivots:

$$\begin{aligned} OP_A &= \Pr(A \text{ wins} | \text{voter } i \text{ votes } A) - \Pr(A \text{ wins} | \text{voter } i \text{ abstains}) \\ &= \frac{1}{2} \Pr(a = b) + \frac{1}{2} \Pr(a = b - 1) = \frac{1}{2}x + \frac{1}{2}y \end{aligned} \quad (1)$$

$$\begin{aligned} OP_B &= \Pr(A \text{ wins} | \text{voter } i \text{ votes } B) - \Pr(A \text{ wins} | \text{voter } i \text{ abstains}) \\ &= -\frac{1}{2} \Pr(a = b + 1) - \frac{1}{2} \Pr(a = b) = -\frac{1}{2}x - \frac{1}{2}v \end{aligned} \quad (2)$$

Guaranteed Prize Pivots:

$$\begin{aligned} GA_{Ak} &= (\text{Group } k \text{'s expected share of } \Gamma_A \text{ if } i \text{ votes for } A) - (k \text{'s expected share of } \Gamma_A \text{ if } i \text{ abstains}) \\ &= \frac{1}{2} \Pr(a_k = \hat{a}_{-k}) + \frac{1}{2} \Pr(a_k = \hat{a}_{-k} - 1) = \frac{1}{2} \cdot \prod_{j=2}^K (w_{kj} + t_{kj} + s_{kj}) - \frac{1}{2} \cdot \prod_{j=2}^K w_{kj} \end{aligned} \quad (3)$$

$$GA_B = (\text{Group } k \text{'s expected share of } \Gamma_A \text{ if } i \text{ votes for } B) - (k \text{'s expected share of } \Gamma_A \text{ if } i \text{ abstains}) = 0 \quad (4)$$

Victory Prize Pivots:

$$\begin{aligned} VA_A &= (\text{Group } k \text{'s expected share of } \Omega_A \text{ if } i \text{ votes for } A) \\ &\quad - (k \text{'s expected share of } \Omega_A \text{ if } i \text{ abstains}) \\ &= \frac{1}{2}(x + y + z) \cdot \prod_{j \neq k} (w_{kj} + t_{kj} + s_{kj}) - \frac{1}{2}z \cdot \prod_{j \neq k} w_{kj} \end{aligned} \quad (5)$$

$$\begin{aligned}
VA_B &= (\text{Group } k \text{'s expected share of } \Omega_A \text{ if } i \text{ votes for } B) - (k \text{'s expected share of } \Omega_A \text{ if } i \text{ abstains}) \\
&= -\frac{1}{2}v \cdot \prod_{j \neq k} (w_{kj}) - \frac{1}{2}(u + v + x) \cdot \prod_{j \neq k} (w_{kj} + t_{kj})
\end{aligned} \tag{8}$$

There are analogous terms, GB_{Ak} , GB_{Bk} , VB_{Ak} and VB_{Bk} , for the prize pivots associated with B 's prizes.

Proof. OP_A is a well-known-result (see for instance, Myerson 1998). Therefore, we focus on the guaranteed prize pivot: GA_{Ak} . If voter i in group k abstains then the probability that her group wins the prize outright is the product of the probabilities that her group has more votes than each of the other groups: $\Pr(W) = \Pr(a_k > a_1 \& a_k > a_2 \& \dots) = \cdot \prod_{j \neq k} w_{kj}$. If i abstains then her group gets no share of the prize if any $a_j > a_k$. This occurs with probability $\Pr(L) = \Pr(a_k < a_1 | a_k < a_2 | \dots) = 1 - \Pr(a_k \geq a_1 \& a_k \geq a_2 \& \dots) = 1 - \cdot \prod_{j \neq k} (w_{kj} + t_{kj})$. Therefore, if i abstains the probability that her group ties for largest number of votes and gets a half prize is $\Pr(tie) = 1 - \Pr(W) - \Pr(L) = \cdot \prod_{j \neq k} (w_{kj} + t_{kj}) - \cdot \prod_{j \neq k} w_{kj}$.

Now suppose voter i votes for party A. Group k wins the prize outright with probability $\Pr(WA) = \Pr(a_k + 1 > a_1 \& a_k + 1 > a_2 \& \dots) = \cdot \prod_{j \neq k} \Pr(a_k + 1 > a_j) = \cdot \prod_{j \neq k} (w_{kj} + t_{kj})$. Group k gets no share of the prize with probability $\Pr(LA) = \Pr(a_k + 1 < a_1 | a_k + 1 < a_2 | \dots) = 1 - \Pr(a_k + 1 \geq a_1 \& a_k + 1 \geq a_2 \& \dots) = 1 - \cdot \prod_{j \neq k} (w_{kj} + t_{kj} + s_{kj})$. Group k is involved in a tie and gets a half share of the prize with probability $\Pr(tieA) = 1 - \Pr(WA) - \Pr(LA) = \cdot \prod_{j \neq k} (w_{kj} + t_{kj} + s_{kj}) - \cdot \prod_{j \neq k} (w_{kj} + t_{kj})$.

The difference between group k 's expected share of the prize Γ_A if i votes for A is therefore $GA_A = \Pr(WA) + \frac{1}{2} \Pr(tieA) - \Pr(W) - \frac{1}{2} \Pr(tie) = \frac{1}{2} \cdot \prod_{j \neq k} (w_{kj} + t_{kj} + s_{kj}) - \frac{1}{2} \cdot \prod_{j \neq k} w_{kj}$.

Since votes for B have no effect on the number of A votes in each group GA_B is zero.

The derivation of the victory prize pivots is slightly more complex but follows a similar form and is shown in the appendix. ■

Voting Calculus

Suppose we consider any fixed vote profile $(p, q) = ((p_1, q_1), (p_2, q_2), \dots, (p_K, q_K))$ that describes the probability with which members of each group support A and B respectively. Given this profile, the following equations characterize the private evaluation of party A relative to B (that is value of ε_i) that would make an individual in group k indifferent between her various vote choices.

$$U_k(\text{voteA}) - U_k(\text{abstain}) = (\gamma + \tau_{Ak})OP_A + \Gamma_A GA_{Ak} + \Omega_A VA_{Ak} + \Omega_B VB_{Ak} - c = 0 \tag{9}$$

$$U_k(\text{voteB}) - U_k(\text{abstain}) = (\gamma + \tau_{Bk})OP_B + \Omega_A VA_{Bk} + \Gamma_B GB_{Bk} + \Omega_B VB_{Bk} - c = 0 \tag{10}$$

$$\begin{aligned}
U_k(\text{voteA}) - U_k(\text{voteB}) &= (\gamma + \tau_{ABk})(OP_A - OP_B) + \\
\Gamma_A(GA_{Ak}) + \Omega_A(VA_{Ak} - VA_{Bk}) + \Gamma_B(-GB_{Bk}) + \Omega_B(VB_{Ak} - VB_{Bk}) &= 0
\end{aligned} \tag{11}$$

The thresholds, τ_{Ak} , τ_{Bk} and τ_{ABk} that solve these equations characterize Nash equilibria.

Theorem 2 *There exist vote profiles supported by Nash equilibrium voting behavior: voter i in group k votes for party A if $\varepsilon_i > \max\{\tau_{Ak}, \tau_{ABk}\}$; votes for B if $\varepsilon_i < \min\{\tau_{Bk}, \tau_{ABk}\}$ and abstains otherwise. The thresholds, τ_{Ak} , τ_{Bk} and τ_{ABk} , solve equations 9, 10, and 11 for each group and $p_k = 1 - F(\max\{\tau_{Ak}, \tau_{ABk}\})$ and $q_k = F(\min\{\tau_{Bk}, \tau_{ABk}\})$.*

Proof. Given the Poisson population assumption, there is always some, be it very small, probability that i is the only voter. In such a setting, her vote would determine the outcome. This ensures that $OP_A > 0$ and $OP_B < 0$. Therefore equation 9 is an increasing linear functions of τ_{Ak} . Therefore for any given vote profile (p, q) , there is a unique threshold that solves the equations (and the same for equations 10, and 11). As annotated, these three equations correspond to differences in expected value from each of the voter's actions. If $\varepsilon_i > \max\{\tau_{Ak}, \tau_{ABk}\}$ then, i votes for A, since $U_k(\text{voteA}) > U_k(\text{abstain})$ and $U_k(\text{voteA}) > U_k(\text{voteB})$. Similarly if $\varepsilon_i < \min\{\tau_{Bk}, \tau_{ABk}\}$, then i votes for B.

Given the thresholds, an individual in group k votes for A with probability $\tilde{p}_k(p, q) = 1 - F(\max\{\tau_{Ak}, \tau_{ABk}\})$ and votes for B with probability $\tilde{q}_k(p, q) = F(\min\{\tau_{Bk}, \tau_{ABk}\})$. Since both outcome and prize pivot probabilities are continuous in all components of the vote profile (p, q) , the τ thresholds, and hence $\tilde{p}_k(p, q)$ and $\tilde{q}_k(p, q)$, are continuous in all components of the vote profile. Let $M : [0, 1]^{2K} \rightarrow [0, 1]^{2K}$ be this best response function for all the groups. That is to say, M maps (p, q) into simultaneous best responses for all groups $(\tilde{p}, \tilde{q}) = ((\tilde{p}_1(p, q), \tilde{q}_1(p, q)), \dots, (\tilde{p}_K(p, q), \tilde{q}_K(p, q)))$. As M is continuous and maps a compact set back into itself, by Brouwer's fixed point theorem (1912), a fixed point exists.¹ ■

Not only do Nash equilibria exist, we can approximate them well in large electorates.

SYMMETRIC EQUILIBRIUM APPROXIMATIONS

As the number of voters grows large, there are simple approximations for the pivot probabilities. We derive these approximations for the symmetric case where the voters in all groups vote for party A with probability p and support party B with probability q . Later we show how these symmetric cases can be adapted to analyze asymmetric cases as well. For what follows we assume N is large and K , the number of groups, is small relative to N .

Lemma 3 *If X and Y are Poisson random variables with means λ and μ , respectively, then $S(\lambda, \mu, m) = \Pr(X - Y = m) = e^{-(\lambda+\mu)} \left(\frac{\lambda}{\mu}\right)^{\frac{m}{2}} I_m(2\sqrt{\lambda\mu})$, where I_m is the modified Bessel function of the first kind. The function S is called the Skellam distribution with parameters λ and μ (See Skellam 1946).*

¹If $F()$ was discontinuous then we would need to introduce mixed strategies for types at the discontinuities and use Kakutani's fixed point theorem.

This lemma provides a convenient means to approximate pivotal probabilities. Additional votes are only influential when they make or break ties. The relevant probabilities are given by the Skellam distribution evaluated around $m = -1, 0, 1$.² The modified Bessel function is a well-known mathematical function, and, as the next lemma describes, provided the expected numbers of voters (that is the λ and μ) are reasonably large, it is well-approximated by a simply exponential expression.

Lemma 4 *If $|m| \leq 1$ and $N\sqrt{pq} > 19.1$ then the following approximation has less than 1% error:*

$$S(Np, Nq, m) \approx \frac{1}{\sqrt[4]{p}\sqrt[4]{q}} \left(\frac{p}{q}\right)^{\frac{m}{2}} \cdot \frac{e^{-N(\sqrt{p}-\sqrt{q})^2}}{2\sqrt{\pi N}} \quad (12)$$

where $p = \frac{1}{N} \sum_{k=1}^K p_k N_k$ and $q = \frac{1}{N} \sum_{k=1}^K q_k N_k$.

If $|m| \leq 1$ and $\frac{Np_k}{K} > 19.1$ then the following approximation has less than 1% error

$$S\left(\frac{Np_k}{K}, \frac{Np_k}{K}, m\right) \approx \frac{1}{2\sqrt{\pi}} \cdot \frac{\sqrt{K}}{\sqrt{p_k N}}. \quad (13)$$

Proof. For fixed m and large x , the Bessel function approximation (Abramowitz and Stegun 1965, p. 377) is

$$I_{|m|}(x) \approx \frac{e^x}{\sqrt{2\pi x}} \left(1 - \frac{4m^2 - 1}{8x} + \frac{(4m^2 - 1)(4m^2 - 9)}{2!(8x)^2} - \frac{(4m^2 - 1)(4m^2 - 9)(4m^2 - 25)}{3!(8x)^3} + \dots\right)$$

We use the first term of this approximation $I_{|m|}(x) \approx \frac{e^x}{\sqrt{2\pi x}}$. To check the accuracy we evaluate $(I_m(x) - \frac{e^x}{\sqrt{2\pi x}})/I_m(x)$ for $m = 0, 1$. Ninety nine percent accuracy is attained when $x = 2\sqrt{\lambda\mu} > 38.2$. The approximation become better as x increases.

We apply this approximation to the Skellam distribution to get

$$\begin{aligned} S(\lambda, \mu, m) &\approx \frac{1}{2\sqrt{\pi}\sqrt[4]{\lambda\mu}} \left(\frac{\lambda}{\mu}\right)^{\frac{m}{2}} e^{-(\lambda+\mu)+2\sqrt{\lambda\mu}} \\ &= \frac{1}{2\sqrt{\pi}\sqrt[4]{\lambda\mu}} \left(\frac{\lambda}{\mu}\right)^{\frac{m}{2}} e^{-(\sqrt{\lambda}-\sqrt{\mu})^2} \end{aligned}$$

Equations 12 and 13 result directly from replacing λ and μ with Np and Nq in 12 and Np_k/K and Np_k/K in 13. ■

Given these lemmas we can derive approximations for the pivot probabilities for large voting populations. We introduce the following notation, $\varphi(K) = 4(K-1)2^K$.

²The approximation of V_{ABk} contains one term evaluated at $m = 2$ but this term is extremely small and so has little impact on vote calculation.

Proposition 5 *If $p_k = p$ and $q_k = q$ for all k , then for large N the following terms approximate pivot probabilities:*

Outcome Pivot Approximations

$$OP_A \approx \frac{1}{2^4 \sqrt{pq} \sqrt{\pi N}} \frac{\sqrt{p} + \sqrt{q}}{2\sqrt{p}} \cdot e^{-N(\sqrt{p}-\sqrt{q})^2} \quad (14)$$

$$OP_B \approx -\frac{1}{2^4 \sqrt{pq} \sqrt{\pi N}} \frac{\sqrt{p} + \sqrt{q}}{2\sqrt{q}} \cdot e^{-N(\sqrt{p}-\sqrt{q})^2} \quad (15)$$

Guaranteed Prize Pivots Approximations

$$GA_{Ak} \approx \varphi(K) \frac{1}{2\sqrt{\pi}} \cdot \frac{\sqrt{K}}{\sqrt{pN}} \quad (16)$$

$$GB_{Bk} \approx \varphi(K) \frac{1}{2\sqrt{\pi}} \cdot \frac{\sqrt{K}}{\sqrt{qN}}, \quad (17)$$

Remember $GA_{Bk} = 0$ and $GB_{Ak} = 0$.

Victory Prize Pivots Approximations

$$VA_{Ak} \approx zGA_A + \frac{OP_A}{2^{K-1}} \quad (18)$$

$$VA_{Bk} \approx \frac{1}{2^{K-1}} \frac{(\sqrt{p} + \sqrt{q})}{\sqrt{q}} OP_B < 0 \quad (19)$$

$$VB_{Bk} = (1 - z - x)GB_B - \frac{OP_B}{2^{K-1}} > 0 \quad (20)$$

$$VB_{Ak} \approx -\frac{1}{2^{K-1}} \frac{(\sqrt{p} + \sqrt{q})}{\sqrt{p}} OP_A < 0 \quad (21)$$

where $z = \Pr(a > b)$ and $x = \Pr(a = b)$

Proof. From

$$\begin{aligned} OP_A &= \frac{1}{2} \mathbb{P}(a = b) + \frac{1}{2} \mathbb{P}(a = b - 1) \\ &= \frac{1}{2} S(Np, Nq, 0) + \frac{1}{2} S(Np, Nq, -1). \end{aligned}$$

Applying the Skellam approximation (12) with the values $m = 0$ and $m = -1$, we have

$$OP_A \approx \frac{1}{2} \left(1 + \frac{q^{1/2}}{p^{1/2}} \right) \cdot \frac{e^{-N(\sqrt{p}-\sqrt{q})^2}}{2^4 \sqrt{pq} \sqrt{\pi N}}.$$

This is the same approximation used by Myerson (1998, p 391). The approximation for $OP_B = -\frac{1}{2}\mathbb{P}(a = b) - \frac{1}{2}\mathbb{P}(a = b + 1)$ is similar.

To calculate the guaranteed prize pivot GA , we first define

$$w = \mathbb{P}(a_1 > a_2), \quad t = \mathbb{P}(a_1 = a_2), \quad \text{and} \quad s = \mathbb{P}(a_1 = a_2 - 1).$$

By the Skellam approximation (13),

$$t = S\left(\frac{Np}{K}, \frac{Np}{K}, 0\right) \approx \frac{1}{2\sqrt{\pi}} \cdot \frac{\sqrt{K}}{\sqrt{pN}}. \quad (22)$$

Since the approximation (13) does not depend on m , we also have that $s = S(\frac{Np}{K}, \frac{Np}{K}, -1) \approx t$. By symmetry, $w = \mathbb{P}(a_1 < a_2)$, hence $2w + t = 1$ and $w = \frac{1-t}{2}$. Combining these facts gives

$$\begin{aligned} GA_{Ak} &= \frac{1}{2}\mathbb{P}(a_1 \geq a_2 - 1)^{K-1} - \frac{1}{2}\mathbb{P}(a_1 \geq a_2 + 1)^{K-1} = \frac{1}{2}(w + t + s)^{K-1} - \frac{1}{2}w^{K-1} \\ &\approx \frac{1}{2}\left(\frac{1-t}{2} + t + t\right)^{K-1} - \frac{1}{2}\left(\frac{1-t}{2}\right)^{K-1} = \frac{1}{2^K}(1 + 3t)^{K-1} - \frac{1}{2^K}(1 - t)^{K-1} \\ &= \frac{1}{2^K}((K-1)(3t - (-t)) + ((3t)^2 - t^2)\frac{(K-1)(K-2)}{2} + ((3t)^3 + t^3)\frac{(K-1)(K-2)(K-3)}{3!} + \dots \\ &\quad + ((3t)^h - (-t)^h)\frac{(K-1)!}{h!(K-1-h)!} + \dots + (3t)^{K-1} - t^{K-1}) \\ &\approx \frac{1}{2^K}4(K-1)t + \frac{1}{2^K}4(K-1)(K-2)t^2 + \frac{1}{2^K}\frac{14}{3}\frac{(K-1)!}{(K-4)!}t^3 \approx \frac{1}{2^K}4(K-1)t \end{aligned}$$

where the last approximation follows because t is small so all higher order powers of t go to zero. However, when K is large and pN relatively small this approximation is relatively inaccurate. This deficiency is readily corrected by adding additional terms. Simplifying, this says

$$GA_{Ak} \approx \varphi(K)t \approx \varphi(K)\frac{1}{2\sqrt{\pi}} \cdot \frac{\sqrt{K}}{\sqrt{pN}}.$$

Substituting (22) for t in this expression, we have proved (16). The formula (17) for GB_{Bk} is similar, except with p 's replaced by q 's. Approximations for the victory prize pivots take a similar form and are shown in the appendix. ■

The pivot approximations conform to intuition. In the symmetric case, the guaranteed prize pivot is effectively the product of the probability of a tie between any two groups ($t = S(\frac{Np}{K}, \frac{Np}{K}, 0)$) and a term reflecting the likelihood that, of all the possible pairings between groups, group k is involved ($\varphi(K) = \frac{K-1}{2^{K-2}}$). The victory prize pivot contains two terms: $VA_{Ak} \approx zGA_{Ak} + \frac{OPA}{2^{K-1}}$. Before group k can win the victory prize it needs to be the most supportive group and party A must win. The first term in the expression for VA_{Ak} represents the probability that a voter is influential in making group k the most supportive (that is simply GA_{Ak}) multiplied by the probability that party A wins. The second term represents the probability that this voter's vote influences the

electoral outcome so that A wins and can allocate the victory prize, weighted by the likelihood that under such a circumstance the prize flows to group k . The difficulty with approximating VA_{Ak} is that it contains the term $z = \mathbb{P}(a > b)$, which is difficult to approximate since it equals the following infinite sum: $z = \sum_{m=1}^{\infty} S(Np, Nq, m)$. However, for the fully symmetric case where $p = q$, $z = (1 - S(Np, Nq, 0))/2$, which is approximately $1/2$ for large N . In this case $VA_{Ak} \approx \frac{GA_{Ak}}{2} + \frac{OP_{Ak}}{2^{K-1}}$. In asymmetric cases where voters are substantially more likely to support A rather than B ($p \gg q$), then party A virtually always wins so z is approximately equal to one. Therefore $VA_{Ak} \approx GA_{Ak}$. That the victory prize pivots are simple extensions of the guaranteed prize pivots means they have similar properties and so in much of what follows we can talk about prizes generally without having to distinguished between the precise type of prize.

For large populations the pivot probability estimates are accurate. For instance, if the population mean is $N = 100,000$, there are $K = 3$ groups and voters support parties A and B with probability $p = .5$ and $q = .5$, then the approximation error is around $.0001\%$ for the outcome pivot and 0.16% for the prize pivot. The error in the prize pivot approximation becomes larger as the number of groups rises, especially if Np is modest in size. To avoid significant underestimation of GA_{Ak} additional terms can be added: $\frac{1}{2^K} 4(K-1)(K-2)t^2$, for instance, is the next term. Since our focus is on how incentives created by prize pivots dominate those from outcome pivots, any underestimation of GA_{Ak} will strengthen, not diminish, our substantive conclusions.

MOTIVATIONS TO VOTE

Using the results above we explore the substantive topics of turnout, vote choice, voter motivation and the rewards parties offer their supporters. Before turning to equilibrium analysis, figure 1 graphs the outcome, guaranteed prize and victory prize pivots as a function of p when the voters are divided into $K = 3$ groups. This figure assumes $N = 100,000$ and $q = .05$. The solid line represents OP_A . As expected it takes its maximal value at $p = q$, which in this case is 5% . At this point approximately 5000 voters vote for A and 5000 vote for B. The chance that an additional vote for either party is influential in determining the electoral outcome is approximately 0.4% . However, except around this point where expected voter support is almost perfectly matched, voters have virtually no influence on the electoral outcome. For instance, in this example if $p = .055$, that is half a percent higher than q , then $OP_A \approx 2.6 \cdot 10^{-8}$.

Figure 1 about here.

In contrast, the guaranteed prize pivot, shown by the dotted line, is a smoothly declining function of p . When turnout is low (small p), casting an additional vote for party A has a strong influence on the allocation of the guaranteed prize. As the number of expected voters for party A increases, the influence of an additional vote diminishes. However, as the figure makes clear, prize pivotalness does not go to zero in the manner that outcome pivotalness does. Referring back to equations 14 and 16 we can see the reason for this. The approximations for OP_A and GA_{Ak} contain the terms

$e^{-N(\sqrt{p}-\sqrt{q})^2} / \sqrt[4]{pq}\sqrt{N}$ and $1/\sqrt{Np}$, respectively. An $N^{-\frac{1}{2}}$ term appears in both expression. As N gets large, this term converges to zero; however it does so at a slow rate. This is the influence N has on the GA_{Ak} term. Outcome pivot also contains an exponential term, $e^{-N(\sqrt{p}-\sqrt{q})^2}$, which converges rapidly to zero as N gets large unless $p = q$. Thus, except around the precise balance position of $p = q$, outcome pivotalness is extremely tiny for even modest sized N .

The victory prize pivot (VA_{Ak}) is plotted with the dashed line. We have displaced this line by adding 0.005 to the VA_{Ak} as otherwise it is difficult to see on the graph. To the right of $p = q$, VA_{Ak} is virtually identical to the guaranteed prize pivot, although as just noted, for clarity it is drawn a little above it. Given the approximation, $VA_{Ak} = zGA_{Ak} + \frac{OP_A}{2K-1}$, this is unsurprising. When $p > q$ then OP_A is extremely small and $z = \Pr(a > b) \approx 1$. To the left of $p = q$, $VA_{Ak} \approx 0$ since $p < q$ means that in a large electorate the chance that party A wins the election is virtually zero. As the figure shows, in passing through the region around $p = q$, the victory prize pivot transitions between being equal to approximately zero and approximately equal to the guaranteed prize pivot.

Although we have not yet moved to an equilibrium analysis, figure 1 provides much of the intuition about voter motivations. The desire to influence which party wins is only a significant motivation for voters when elections are close. Rewards that a party can give their most supportive groups (guaranteed prizes) provide voters with a motive to turnout. Competition between groups to receive the rewards that a governing party can hand out if they win office (that is a victory prize) provides a strong motivation to support that party, but only once that party is likely to win office.

Given these intuitions we now examine a series of equilibria to see how these incentives play out in influencing equilibrium voting behavior.

Fully Symmetric Equilibria

If there is no policy bias in favor of either party ($\gamma = 0$) and both parties offer the same prizes ($\Gamma_A = \Gamma_B = \Gamma$ and $\Omega_A = \Omega_B = \Omega$), then there is an equilibrium where all groups vote for parties A and B at the same rate ($p_k = q_k = p$ for all k) and, assuming a sufficiently high cost of voting, p solves

$$U_{i \in k}(\text{vote } A) - U_{i \in k}(\text{abstain}) = (F^{-1}(1-p))OP_A + \Gamma_A GA_{Ak} + \Omega_A VA_{Ak} + \Omega_B VB_{Ak} - c = 0,$$

which for large N is well approximated by the solution to $(F^{-1}(1-p))\frac{1}{2\sqrt{p}\sqrt{\pi N}} + (\Gamma + \frac{\Omega}{2})\varphi(K)\frac{1}{2\sqrt{\pi}}\frac{\sqrt{K}}{\sqrt{pN}} - c = 0$.

These claims follow directly from the propositions above. If the cost of voting is very low then $p = q = 1/2$ solves equation 11 instead. Figure 2 plots voter turnout for each party as a function of the size of prizes $((\Gamma + \frac{\Omega}{2}))$ for this equilibrium assuming the cost of voting is $c = .01$. When the parties offer few prizes, turnout is low. For instance when there are no prizes then in this symmetric case turnout is only about 3.6%, with half supporting A and the other half supporting B.

Figure 2 about here.

This low level of turnout forms the basis of the standard critique of rational choice voting models.

Since voters have little influence on the outcome, even in this evenly balanced situation, only those voters who have a strong (either positive or negative) evaluation of party A vote. However, if voters are also competing to win prizes for their groups, then turnout increases as the value of these prizes rises. For instance, if the prize size rises to $(\Gamma + \frac{\Omega}{2}) = 2$ relative to the cost of voting of $c = .01$ then turnout increases to 28% (14% for each party) if there are $K = 3$ groups. In contrast, the effect is more measured when the number of groups is larger. For instance, if there are 7 groups then increasing prizes to $(\Gamma + \frac{\Omega}{2}) = 2$ only results in a turnout of about 8%. Of course, these calculations require some calibration between the cost of voting, in this case assumed to be $c = .01$, and the value of prizes. The comparisons above assumed the prizes were worth $(\Gamma + \frac{\Omega}{2}) = 2$, that is 200 times the cost of voting. If we take the value of selective rewards to be on this scale, then the distribution of contingent prizes provides an explanation of significantly higher turnout than predicted by the standard pivotal voting model where voters care only about which party wins.

As figure 2 graphically demonstrates, the number of groups affects the extent to which voters are motivated to compete for prizes. The number of groups, K , has no influence on outcome pivot, but it strongly influences the prize pivotalness. This is most easily seen by examining the approximation for the prize pivots. Both the guaranteed and victory prize pivots approximations contain a $\varphi(K)\sqrt{K}$ term, which takes its maximal value at $K = 3$. Thus if prizes are pure local public goods, then turnout is maximized by 3 groups competing to be the most supportive.

An alternative reasonable assumption is that the value of the prize is inversely proportional to the number of people receiving it. This would be the case if rewards take the form of transfers or the chance of a job for members of the group. In this setting as the number of groups rise, the number of people in each group diminishes and each group member's share of any prize increases. In this formulation the value of prizes increases linearly in K , and so $\Gamma_A G A_A$ and $\Omega_A V A_A$ both vary as $\varphi(K)\sqrt{K}K$. Under this situation prize pivotality is maximized at $K = 4$ groups. Whether prizes are conceptualized as pure local public goods or private goods dispensed among members of the group, it is clear that a relatively small number of groups is most effective at inducing turnout in a highly competitive election.

Parties induce political loyalty by giving selective benefits to supportive groups. This mechanism works best when the number of groups is relatively small. Placing voters in clearly demarcated groups in a large national electorate might be difficult. The problem of motivating voters might be simplified if the voters are divided into collections of groups. The following example provides a simple illustration of the trade-offs. Suppose a constituency is composed of 27 villages. A party could take three approaches to inducing political support. First, it could offer a single prize of size 27 to the most supportive of the 27 villages. Alternatively, the party could merge villages to form three groups of nine villages and offer a prize of size 3 to each of the villages in the most supportive collection of villages (that is the prize of size 27 is split amongst the 9 villages in the most supportive group). Finally the party might assemble neighboring villages into clusters of 3 villages and offer to give a prize of size 3 to the most supportive village in each cluster.

To gain some intuition as to which scheme induces the most political support, and therefore the highest turnout, it is worthwhile comparing the product of the prize size and prize pivot under each contingency ($\Gamma_A G A_{Ak}$). The product of pivot probability and prize size is orders of magnitude larger for the latter two configurations than the former. A large number of small groups competing at the constituency level for a large prize induces relatively little political loyalty.³ The product of prize size and prize pivot is three times larger in the case of villages clustered into threes than the formation of 3 constituency level groups. This example is constructed assuming prizes are private in nature and hence scale linearly. If there are returns to scale in the allocation of prizes then the balance is pushed back towards 3 constituency level groups being the most efficient means to induce turnout.

Even in the highly competitive case of the fully symmetric equilibrium where both parties are evenly matched, voters are as much motivated by selective group rewards as by a desire to influence the electoral outcome. And in this setting parties garner greater political support if they can divide voters into a small number of constituency-level groups or if they dispense prizes locally within clusters of small groups. The competition for prizes is an important political consideration even in highly competitive elections. As we see in the next case, competition for prizes becomes virtually the only motivation to turnout and vote when elections are lopsided.

Dominant Party Equilibria

Magaloni (2006) argues Mexico's PRI remained a dominant party long after it was generally acknowledged by virtually all voters to be the inferior choice. This is not an isolated example. Numerous books have been dedicated to characterizing dominant parties (see, for example, Simkins 1999; Sartori 1976). Obviously in some cases there is substantial voter intimidation and fraud, but even excluding these, in many countries the incumbent party appears to win election after election even though it provides few benefits. Patronage is often cited as the reason for such party dominance. Yet, as reviewed above, this classic vote buying approach is problematic.

Contingent prize allocations incentivize voters to support parties even if they dislike the party. In doing so this approach describes a patronage like system, but without encountering credibility problems or requiring many people to receive rewards. Once one party is perceived as dominant and virtually certain to win the election, policy preferences have little influence on voting behavior. To see why this is so, we consider the case where p is larger than q and examine the decision to vote for A rather than abstain as characterized by equation 9. Given that $p > q$ for large N , $OP_A \approx \frac{1}{2} \left(1 + \frac{q^{1/2}}{p^{1/2}}\right) \cdot \frac{e^{-N(\sqrt{p}-\sqrt{q})^2}}{2^{4/pq}\sqrt{\pi N}} \approx 0$ so a voter's preference for party A over party B has virtually no influence in her voting calculus. As clearly shown by figure 1, voters have a negligible influence on the electoral outcome unless the expected vote shares balance. The guaranteed and victory prize

³The comparisons of $\Gamma_A G A_{Ak} \approx \Gamma_A \frac{K-1}{2^{K-2}} \frac{1}{2\sqrt{\pi}} \frac{\sqrt{K}}{\sqrt{pN}}$ are $27 \frac{27-1}{2^{27-2}} \frac{1}{2\sqrt{\pi}} \frac{\sqrt{27}}{\sqrt{pN}}$, $3 \frac{3-1}{2^{3-2}} \frac{1}{2\sqrt{\pi}} \frac{\sqrt{3}}{\sqrt{pN}}$ and $3 \frac{3-1}{2^{3-2}} \frac{1}{2\sqrt{\pi}} \frac{\sqrt{3}}{\sqrt{p(N/9)}}$. Given the large number of groups in the first case, the approximation of $G A_{Ak}$ (equation 3) underestimates the prize pivot by about a third if $p = .1$.

pivots for party A both approximately equal $\varphi(K) \frac{1}{2\sqrt{\pi}} \frac{\sqrt{K}}{\sqrt{pN}}$. Substituting these approximations into equation 9 yields $p \approx \left(\varphi(K) \frac{1}{2\sqrt{\pi}} \frac{\sqrt{K}}{\sqrt{N}} \frac{(\Gamma_A + \Omega_A)}{c} \right)^2$. Given that B is extremely unlikely to win the election, a voter's calculus between voting for B or abstaining (equation 10) reduces to approximately $\Gamma_B G B_{Bk} - c = 0$; so $q \approx \left(\varphi(K) \frac{1}{2\sqrt{\pi}} \frac{\sqrt{K}}{\sqrt{N}} \frac{\Gamma_B}{c} \right)^2$. Provided that $\Gamma_A + \Omega_A$ is bigger than Γ_B , this justifies the assumption that $p > q$.

Figure 3 plots the probability of voting for parties A and B, that is p and q , when there are $K = 5$ groups, $N = 100,000$, $\Gamma_A + \Omega_A = 3$ and $\Gamma_B = 1$. As the figure shows, voters disproportionately support party A so voters have virtually no chance of influencing the electoral result. Voters support A or B in order to enhance the prospects of winning prizes for their group. Since party A is virtually certain to win, it can offer both the guaranteed prize and the victory prize as enticements to voters. Realistically, a vote for B is only likely to influence the distribution of B's guaranteed prize. Since A offers greater prizes, it attracts more supporters. The ratio of $\frac{p}{q}$ is approximated by $\frac{(\Gamma_A + \Omega_A)^2}{(\Gamma_B)^2}$, which equals 9 in the example shown in figure 3. As is to be expected, the cost of voting affects turnout. In figure 3, if the cost of voting falls to $c = .0032$, then there is full turnout. As costs increases a smaller proportion of voters participate, but those that do disproportionately support A.

Figure 3 about here

Above we considered a simple example of how a party might optimally aggregate 27 villages into groups. Is it better for a dominant party to have 27 individual villages compete for a large prize, aggregate villages into 3 groups of nine villages or cluster villages and let the villages compete for a small prize within each cluster? We now return to that aggregation problem. Suppose party A is a dominant party and it sets a goal of getting 60% of the voters to turnout and vote for it. We pose the following question, if the expected population is $N = 100,000$ and voting has a cost of $c = .1$, then how large a prize would A need to offer to induce its desired 60% support?

If A offers a single prize to the most supportive of 27 villages⁴, then to induce 60% support the prize would need to be $\Gamma_A + \Omega_A \approx 2 \cdot 10^7$. If party A aggregates villages to form three groups of nine villages then it would need to offer a prize worth $\Gamma_A + \Omega_A \approx 50$ to the nine villages in the winning group to induce $p = .6$. Finally, if A organizes the villages into clusters, each of which contained three villages, then it would need to offer prizes worth $\Gamma_A + \Omega_A \approx 17$ to the most supportive village in each of the nine clusters to obtain 60% support. Without a theory of the returns to scale, the public versus private nature of rewards and feasibility of different group structures, it is difficult to definitively decide which of the latter two schemes empowers A most effectively. However, it is clear that both are vastly more effective than the former. Dominant parties most effectively elicit political support by either inducing competition between a small number of national groups, such as major ethnic divisions, or creating competition for resources between local groups, such as snow plowing in Chicago's wards.

⁴In making this calculation we added additional terms in the approximation for $G A_A$, since as noted, the approximation in equation 3 becomes less accurate as K increases.

Dominant party equilibria do not depend upon the popularity of parties. This is to say, γ , the average assessment of A relative to B could be extremely negative and still the voting pattern seen in figure 3 persists. In the dominant party setting, voters have only a negligible influence over which party wins. Unless A's popularity, γ , is on the scale of $1/OP_A \approx \sqrt{N}e^{N(\sqrt{p}-\sqrt{q})^2}$ policy considerations exert little influence on voting decisions. As can be seen in figure 1, since OP_A is virtually zero everywhere except $p = q$, A's unpopularity would need to be colossal to unravel the dominant party equilibrium.

One of the appealing features of the contingent prize allocation mechanism is that it does not encounter the credibility issues that undermine direct vote buying mechanisms. Provided the winning party is willing to allocate a prize to the most supportive group, the equilibrium is stable. Voters are individually incentivized to support the winning party, even if they intensely dislike it. They do so because they want their group to win prizes and the probability that their vote influences the distribution of prizes is sufficiently high that it covers their cost of voting. Traditional vote buying mechanisms have credibility issues. If parties cannot monitor individual votes, then once in the polling booth voters are free to vote for any party whether they took a bribe or not. What is more, parties rarely buy enough votes to account for the support they receive.

Dominant party A virtually always wins and rewards its most supportive groups. This equilibrium analysis assumes all groups support A at the same rate. Of course, it is plausible that groups that receive rewards from one party might gravitate towards supporting that party and those receiving the smaller guaranteed prizes from the opposition become opposition strongholds. Next we explore the equilibrium consequences of such polarization.

Group Polarization

Huckfeldt and Sprague (1995) claim socialization is an important component of how people vote. They argue people adopt the values of their neighbors and so eventually neighbors end up voting for similar candidates. Our model provides an alternative explanation for the convergence of vote choices within groups because voting the same way as your fellow group members (neighbors, in this case) maximizes your group's likelihood of being rewarded by a political party.

Equilibrium behavior supports the endogenous polarization of groups into pro-government and pro-opposition supporters. Suppose that voters in $K_A \geq 2$ of the K groups predominately vote for party A, while the voters in the remaining K_B groups predominately support party B. In particular the voters in the pro-A groups vote for A with probability $p_A \approx \left(\varphi(K_A) \frac{1}{2\sqrt{\pi}} \frac{\sqrt{K_A} (\Gamma_A + \Omega_A)}{\sqrt{N_A} c} \right)^2$, where $N_A = N K_A / K$ and virtually never support party B, $q_A \approx 0$ and the voting behavior in the K_B groups is to vote for party B with probability $q_B \approx \left(\varphi(K_B) \frac{1}{2\sqrt{\pi}} \frac{\sqrt{K_B} (\Gamma_B)}{\sqrt{N_B} c} \right)^2$, where $N_B = N K_B / K$, and virtually never vote for A, $p_B \approx 0$. Provided that $\frac{N}{K} K_A p_A$ is substantially greater than $\frac{N}{K} K_B q_B$ then an equilibrium of this form exists. Figure 4 illustrates just such an equilibrium with $K_A = 3$ and $K_B = 2$ for the same conditions considered in figure 3.

Figure 4 about here

Since $\frac{N}{K}K_A p_A \gg \frac{N}{K}K_B q_B$, party A is virtually certain to win the election and so the outcome pivots are negligible and the voters are motivated predominantly by the prospect of prizes. Voters in the pro-A groups have little incentive to vote for party B. They have a negligible chance of altering the electoral outcome and their group has virtually no chance of winning prizes from party B. Therefore, voters in pro-A groups are almost exclusively motivated by the prospects of capturing the prizes that party A hands out. The decision calculus of pro-A voters equates the cost of voting to the probability of being prize pivotal multiplied by the value of the prizes party A distributes. Thus p_A is calculated by reformulating $\Gamma_A G A_{Ak} + \Omega_A V A_{Ak} - c = 0$ calculated on the subpopulation of K_A groups and a population with mean N_A . The equilibrium level of support for party B is obtained in a similar way, although since B virtually always loses only the guaranteed prize is relevant. Since A virtually always wins outcome and victory pivots are approximately zero in a large population so q_B approximately solves $\varphi(K_B) \frac{1}{2\sqrt{\pi}} \frac{\sqrt{K_B}}{\sqrt{q} N_B} \Gamma_B = c$, that is the guaranteed prize pivot for party B calculated on the set of pro-B groups equates to the voting cost-prize ratio (c/Γ_B).

As seen in the previous asymmetric case, this equilibrium is robust and stable to the policy preferences of voters. Voters in the pro-A groups might intensely dislike A ($\gamma \ll 0$), but since A virtually always wins the election this dislike plays only an infinitesimal role in the decision calculus of voters. Instead voters in pro-A groups are motivated to support A in the hope of winning the prize for their group. The competition for prizes between groups drives voting behavior.

Before moving on, it is worth exploring some important insights about the number of pro-A and pro-B groups. Perhaps surprisingly the number of pro-A groups can be less than the number of pro-B groups. That is dominant parties can maintain their status based on support from a minority of the population. Since party A is anticipated to win, groups supporting A compete for both guaranteed and victory prizes. In contrast, party B is expected to distribute only the smaller guaranteed prize. With greater prizes to allocate, party A can elicit greater turnout from supporters in its smaller number of aligned groups than can party B from a great number of groups.

Another interesting point is that the dominant party must have at least two groups vying for the prizes it offers. To see why, consider the contradiction and suppose party A had only one group of supporters. Voters in this group know there is no rival group competing for the prize so they have no incentive to vote for it ($G A_A \approx 0$ and $V A_A \approx 0$). This results in a low turnout within this group, which contradicts A being a dominant party. Parties garner greater support when there are rival groups competing for any prize they can offer.

Differential Motivation and Turnout

Voters are motivated to turnout by the prospects of altering the electoral outcome in favor of their preferred party and obtaining selective rewards for their group. So far we have seen that even in competitive elections the competition for prizes tends to dominate policy considerations. In this

final illustration we show that prizes discourage policy-based voting even among voters whose group is unlikely to receive any prizes. Suppose there are 9 groups: 3 pro-A groups ($K_A = 3$), 3 pro-B groups ($K_B = 3$) and 3 moderate groups. Let p_A, q_A, p_B, q_B, p_M and q_M represent the probability that voters in each of these sets of groups supports A and B. We examine a symmetric case where both parties offer the same sized prizes ($\Gamma_A = \Gamma_B$ and $\Omega_A = \Omega_B$) and there is no policy bias in favor of either party ($\gamma = 0$). Provided that p_A is substantially greater than p_M and p_B , prizes allocated by party A are virtually certain to go to one of the pro-A groups. Similarly, one of the pro-B groups almost certainly captures the prizes allocated by B. The moderate groups have virtually no chance of receiving any prize (so for these moderate groups, $GA_A \approx GB_B \approx VA_A \approx VA_B \approx 0$) and therefore their vote calculus effectively reduces to equating the outcome pivot with the cost of voting (for these moderate groups: $F^{-1}(1 - p_M)OP_A = c = -F^{-1}(q_M)OP_B$).

Figure 5 about here.

In contrast, the pro-A and pro-B groups are motivated by both these policy incentives and the competition for prizes. As the value of prizes increases, turnout increases in the pro-A and pro-B groups. In contrast, turnout in the moderate groups, where voters vote only with regards to policy considerations, is suppressed by prizes. Figure 5 plots $p_A = q_B$ and $p_M = q_M \approx q_A = p_B$ against prize size $(\Gamma_A + \Omega_A/2)$ for the case where $c = .01$. When there are no prizes, all groups behave identically, with all voters having about 1.8% chance of voting for each party. However, as the size of the prizes increases, the pro-A and pro-B groups are motivated to turnout at a higher rate in order to influence the distribution of the prizes. This increased turnout by the polarized voters reduces turnout by the moderates.

CONCLUSIONS

A central puzzle of democratic politics is why anyone votes when the electorate is large and, therefore, each individual voter's chance of influencing the outcome is near zero. Critics of rational choice theorizing point to voter turnout as a contradiction of rational action when voting is even slightly costly. We present a model in which some prospective voters abstain, others turnout to vote, and we show how turnout varies with changes in the costs of voting and other important characteristics of elections. The model examines the incentives to vote on the basis of policy preferences, partisanship, and in pursuit of what we call contingent prizes. The contingent prizes are assessed both as local public goods distributed to identifiable groups and as individual private benefits allocated within identifiable groups. We show that political parties have incentives to create small numbers of voter blocs and that voters, likewise, have incentives to organize themselves into identifiable groups and that these incentives are largely driven by the value of contingent prizes and their allocation rule. A central result of the model demonstrates that even though voters in large electorates have little influence over which party wins, they retain substantial influence over the distribution of rewards and benefits if parties disproportionately reward groups that provide them

with high levels of political support. Our analysis suggests group-level rewards provide a stronger impetus for voting than do policy differences across parties or other sources of partisanship. What is more, since voters are shown generally to compete over prizes and not policy, our model predicts non-trivial turnout even in cases where one party is almost certain to win.

Pork barrel rewards to local constituencies or preferential benefits, like tax breaks or government jobs, to specific groups are a persistent feature of politics. The model endogenizes the use of these political tools to disproportionately reward identifiable groups of voters. In doing so it provides an explanation for the aggregation, for instance, of Congressional election votes by precinct even though the electoral outcome is unrelated to winning any number of individual precincts. Of course, if votes were put in a common pool and added up at the district level then it would not be possible for parties to identify their most supportive voter groups, eradicating the contingent prize incentive for voting. That incentive serves the interests both of parties and of voters. According to the model, when parties offer few prizes, turnout is low, with only voters with a strong preference for one or the other party voting. Once contingent prizes are thrown into the mix, then voters compete to win benefits for their group and turnout increases as the value of the prizes increases. Indeed, even when the electoral outcome is a forgone conclusion, the value of the contingent prize can induce voters to turnout even to support a party whose policies they oppose. Hence, we show that the strong incentive to pursue local group benefits is an explanation for the persistence of dominant parties even when the electorate acknowledges that an alternative party's policies are preferable.

The model shows that when a party is interested in winning with high turnout, it is better off having even its own supporters divided into at least two discernible groups rather than being one hegemonic bloc. Two groups foster competition over contingent benefits and, therefore, induce higher turnout which can then be used to claim an electoral mandate. Voters share the desire to organize into groups but their interests could be served by forming more groups than is optimal from a party's perspective.

As with any model, there are, of course, limitations. The model points to the necessity of developing a theory to explain when prizes take the form of local public goods and when they take the form of group private goods. Group formation may be driven by political elites or by local voter-entrepreneurs. Depending on which plays the main role in bloc formation there will be variation in the number of groups, the size of prizes, and probably the form of prizes. We have not addressed these issues except by assumption or by example. Then, of course, there is the empirical challenge of testing the observable conditions of turnout, pork allocation and the like under the model's equilibrium conditions. All of these issues remain for future research.

APPENDIX

Proposition 1: Victory Prize Pivot.

$$VA_A = \frac{1}{2}(x + y + z) \cdot \prod_{j \neq k} (w_{kj} + t_{kj} + s_{kj}) - \frac{1}{2}z \cdot \prod_{j \neq k} w_{kj}$$

$$VA_B = -\frac{1}{2}v \cdot \prod_{j \neq k}(w_{kj}) - \frac{1}{2}(u + v + x) \cdot \prod_{j \neq k}(w_{kj} + t_{kj}).$$

Proof. Consider how voter i in group 1 affects victory prize distribution. First suppose i abstains. Let $\Pr(W)$, $\Pr(L)$ and $\Pr(tie)$ represent the probability group 1 gets a full share, no share and a half share of the victory prize.

$$\Pr(W) = \Pr(a_k > a_1 \& a_k > a_2 \& \dots \& \Sigma a > \Sigma b) = z \cdot \prod_{j \neq k} w_{kj}.$$

$$\Pr(L) = \Pr(a_k < a_1 | a_k < a_2 | \dots | \Sigma a < \Sigma b) = 1 - \Pr(a_k \geq a_1 \& a_k \geq a_2 \& \dots \& \Sigma a \geq \Sigma b) = 1 - (x + z) \cdot \prod_{j \neq k}(w_{kj} + t_{kj})$$

$$\text{Therefore } \Pr(tie) = 1 - \Pr(W) - \Pr(L) = (z + x) \cdot \prod_{j \neq k}(w_{kj} + t_{kj}) - z \cdot \prod_{j \neq k} w_{kj}.$$

Next suppose i votes for A. Let $\Pr(WA)$, $\Pr(LA)$ and $\Pr(tieA)$ represent the probability group k gets a full share, no share and a half share of the victory prize, respectively, under this contingency.

$$\Pr(WA) = \Pr(a_k + 1 > a_1 \& a_k + 1 > a_2 \& \dots \& \Sigma a + 1 > \Sigma b) = (x + z) \cdot \prod_{j \neq k}(w_{kj} + t_{kj})$$

$$\Pr(LA) = \Pr(a_k + 1 < a_1 | a_k + 1 < a_2 | \dots | \Sigma a + 1 < \Sigma b) = 1 - \Pr(a_k + 1 \geq a_1 \& a_k + 1 \geq a_2 \& \dots \& \Sigma a + 1 \geq \Sigma b) = 1 - (x + y + z) \cdot \prod_{j \neq k}(w_{kj} + t_{kj} + s_{kj})$$

$$\text{Therefore, } \Pr(tieA) = 1 - \Pr(WA) - \Pr(LA) = (x + y + z) \cdot \prod_{j \neq k}(w_{kj} + t_{kj} + s_{kj}) - (x + z) \cdot \prod_{j \neq k}(w_{kj} + t_{kj}).$$

$$\text{If voter } i \text{ supports party B then } \Pr(WB) = \Pr(a_k > a_1 \& a_k > a_2 \& \dots \& \Sigma a > \Sigma b + 1) = (z - v) \cdot \prod_{j \neq k}(w_{kj})$$

$$\Pr(LB) = \Pr(a_k < a_1 | a_k < a_2 | \dots | \Sigma a < \Sigma b + 1) = 1 - \Pr(a_k \geq a_1 \& a_k \geq a_2 \& \dots \& \Sigma a \geq \Sigma b + 1) = 1 - (z - u - v) \cdot \prod_{j \neq k}(w_{kj} + t_{kj}).$$

$$\text{Therefore, } \Pr(tieB) = 1 - \Pr(WB) - \Pr(LB) = (z - u - v) \cdot \prod_{j \neq k}(w_{kj} + t_{kj}) - (z - v) \cdot \prod_{j \neq k}(w_{kj})$$

Group 1's expected shares of the victory prize if i abstains, votes for A and votes for B are therefore $\Pr(W) + \frac{1}{2}\Pr(tie)$, $\Pr(WA) + \frac{1}{2}\Pr(tieA)$ and $\Pr(WB) + \frac{1}{2}\Pr(tieB)$, respectively.

Hence by voting for A rather than abstaining i increases group 1's expected share of the victory prize by $VA_A =$

$$\begin{aligned} \Pr(WA) + \frac{1}{2}\Pr(tieA) - \Pr(W) - \frac{1}{2}\Pr(tie) &= (x + z) \cdot \prod_{j \neq k}(w_{kj} + t_{kj}) + \frac{1}{2}(x + y + z) \cdot \prod_{j \neq k}(w_{kj} + t_{kj} + s_{kj}) \\ &- \frac{1}{2}(x + z) \cdot \prod_{j \neq k}(w_{kj} + t_{kj}) - z \cdot \prod_{j \neq k} w_{kj} - \frac{1}{2}(z + x) \cdot \prod_{j \neq k}(w_{kj} + t_{kj}) + \frac{1}{2}z \cdot \prod_{j \neq k} w_{kj} \\ &= \frac{1}{2}(x + y + z) \cdot \prod_{j \neq k}(w_{kj} + t_{kj} + s_{kj}) - \frac{1}{2}z \cdot \prod_{j \neq k} w_{kj} \end{aligned}$$

Similarly, by voting for B rather than abstaining, voter i reduces groups 1's expected share of the victory prize, Ω_A , by $VA_B = \Pr(WB) + \frac{1}{2}\Pr(tieB) - \Pr(W) - \frac{1}{2}\Pr(tie) = (z - v) \cdot \prod_{j \neq k}(w_{kj}) + \frac{1}{2}(z - u - v) \cdot \prod_{j \neq k}(w_{kj} + t_{kj}) - \frac{1}{2}(z - v) \cdot \prod_{j \neq k}(w_{kj}) - z \cdot \prod_{j \neq k} w_{kj} - \frac{1}{2}(z + x) \cdot \prod_{j \neq k}(w_{kj} + t_{kj}) + \frac{1}{2}z \cdot \prod_{j \neq k} w_{kj} = -\frac{1}{2}v \cdot \prod_{j \neq k}(w_{kj}) - \frac{1}{2}(u + v + x) \cdot \prod_{j \neq k}(w_{kj} + t_{kj})$. ■

Approximation of the Victory Prize Pivot

To calculate the victory prize pivot VA_{Ak} , we use the quantities t , $w = \frac{1-t}{2}$, and $s \approx t$ as in the GA_A case.

$$VA_A = \frac{1}{2}(x + y + z) \cdot \prod_{i=2}^K(w_{1i} + t_{1i} + s_{1i}) - \frac{1}{2}z \cdot \prod_{i=2}^K w_{1i} = \frac{1}{2}(x + y + z)(w + t + s)^{K-1} - \frac{1}{2}z(w)^{K-1}$$

where $z = \mathbb{P}(a > b)$, $x = \mathbb{P}(a = b)$ and $y = \mathbb{P}(a = b - 1)$.

$$\begin{aligned}
VA_A &= \frac{1}{2}(w+t+s)^{K-1}(x+y+z) - \frac{1}{2}w^{K-1}z \\
&\approx \frac{1}{2}\left(\frac{1-t}{2} + t + t\right)^{K-1}(x+y+z) - \frac{1}{2}\left(\frac{1-t}{2}\right)^{K-1}z \\
&= \frac{1}{2^K}(1+3t)^{K-1}(x+y+z) - \frac{1}{2^K}(1-t)^{K-1}z \\
&= \frac{z}{2^K}[(1+3t)^{K-1} - (1-t)^{K-1}] + \frac{(x+y)}{2^K}(1+3t)^{K-1} \tag{23}
\end{aligned}$$

$$\begin{aligned}
&\approx \frac{z}{2^{K-2}}(K-1)t + \frac{(x+y)}{2^K} + \frac{(x+y)}{2^K}3t(K-1) \\
&\approx \frac{z}{2^{K-2}}(K-1)t + \frac{(x+y)}{2^K} \tag{24}
\end{aligned}$$

$$= zGA_A + \frac{OP_A}{2^{K-1}} \tag{25}$$

since any product of t , x and y is approximately 0.

Now we calculate VA_B . Let

$$u = \mathbb{P}(a = b + 2) \quad \text{and} \quad v = \mathbb{P}(a = b + 1).$$

$$\text{From above } VA_B = -\frac{1}{2}v \cdot \prod_{i=2}^K(w_{1i}) - \frac{1}{2}(u+v+x) \cdot \prod_{i=2}^K(w_{1i} + t_{1i})$$

$$\begin{aligned}
VA_B &= -\frac{1}{2}vw^{K-1} - \frac{1}{2}(u+v+x)(w+t)^{K-1} \\
&= -\frac{1}{2}v\left(\frac{1-t}{2}\right)^{K-1} - \frac{1}{2}(u+v+x)\left(\frac{1-t}{2} + t\right)^{K-1} \\
&= -\frac{1}{2^K}(v(1-t)^{K-1} + (u+v+x)(1+t)^{K-1}) \\
&\approx -\frac{1}{2^K}(v + (u+v+x)) \\
&= -\frac{1}{2^K}(u + 2v + x)
\end{aligned}$$

since any product of u , v , x and t is approximately zero. Applying the Skellam approximation (13), this is

$$\begin{aligned}
VA_B &\approx -\frac{1}{2^K}(S(Np, Nq, 2) + 2S(Np, Nq, 1) + S(Np, Nq, 0)) \\
&\approx -\frac{1}{2^K}\left(1 + 2\frac{\sqrt{p}}{\sqrt{q}} + \frac{p}{q}\right) \cdot \frac{e^{-N(\sqrt{p}-\sqrt{q})^2}}{\sqrt{pq}2\sqrt{\pi N}} = -\frac{1}{2^K}\left(\frac{q + 2\sqrt{pq} + p}{q}\right) \cdot \frac{e^{-N(\sqrt{p}-\sqrt{q})^2}}{\sqrt{pq}2\sqrt{\pi N}} \\
&= -\frac{1}{2^K}\left(\frac{(\sqrt{q} + \sqrt{p})^2}{q}\right) \cdot \frac{e^{-N(\sqrt{p}-\sqrt{q})^2}}{\sqrt{pq}2\sqrt{\pi N}} = \frac{1}{2^{K-1}}\frac{(\sqrt{p} + \sqrt{q})}{\sqrt{q}}OP_B
\end{aligned}$$

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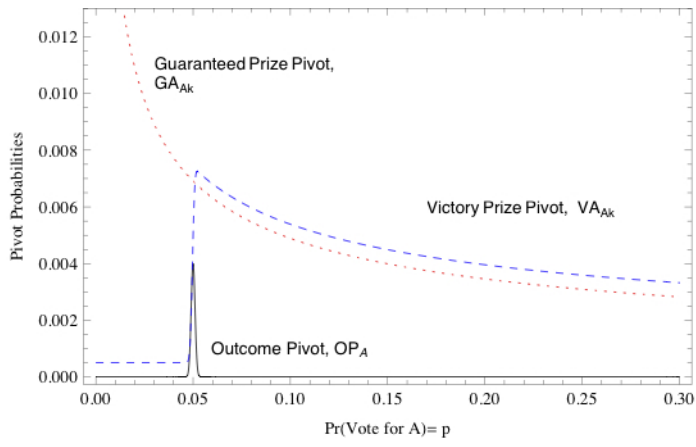


FIG. 1. Pivot Probabilities

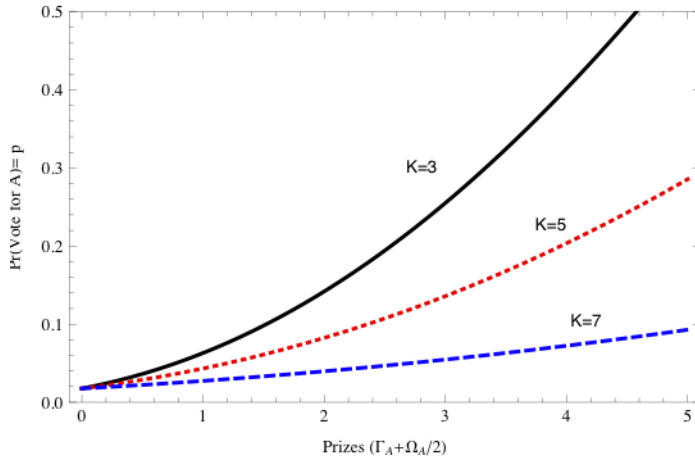


FIG. 2. Turnout and Prizes in the Fully Symmetric Equilibrium

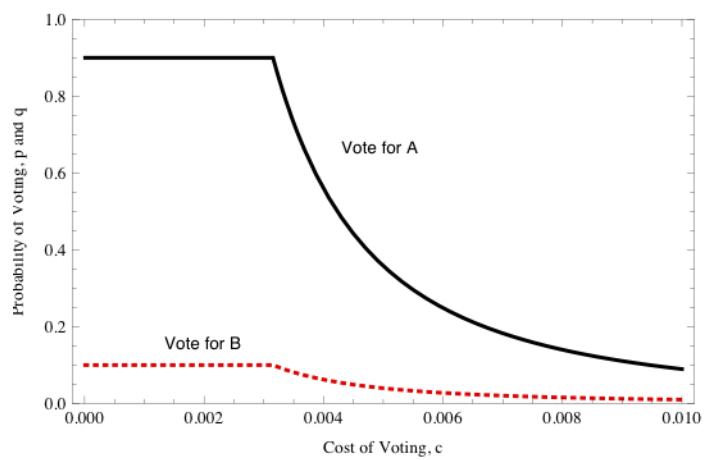


FIG. 3. Voting in Dominant Party Equilibrium

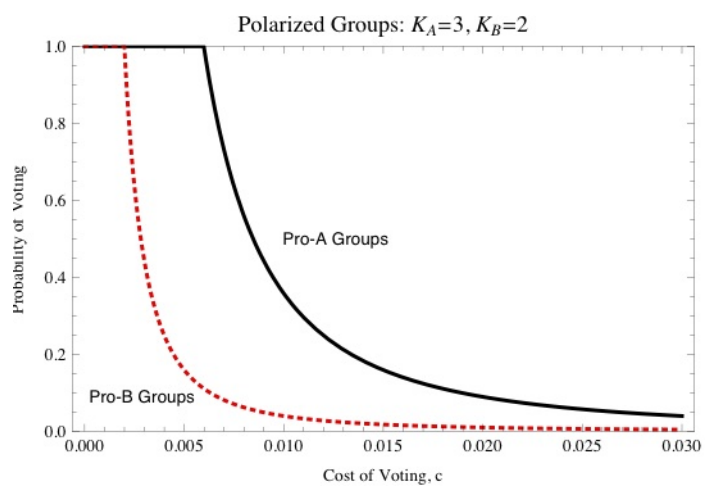


FIG. 4. Pro-A and Pro-B Groups

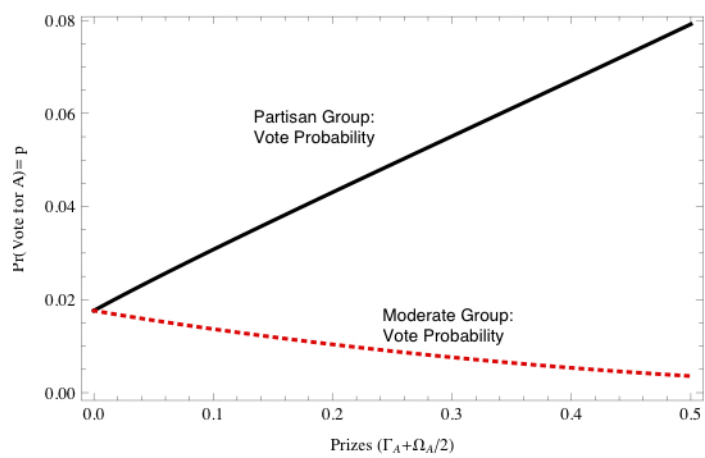


FIG. 5. Differential Vote Motivations